Strong coupling constant from τ decay within a renormalization scheme invariant treatment

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We extract a numerical value for the strong coupling constant α_s from the τ-lepton decay rate into nonstrange particles. A new feature of our procedure is the explicit use of renormalization scheme invariance in analytical form in order to perform the actual analysis in a particular renormalization scheme. For the reference coupling constant in the $\overline{\rm MS}$-scheme we obtain $\alpha_s(M_\tau) = 0.3184 \pm 0.0060_{exp}$ which corresponds to $\alpha_s(M_Z) = 0.1184 \pm 0.0007_{exp} \pm 0.0006_{hq\ mass}$. This new numerical value is smaller than the standard value from τ-data quoted in the literature and is closer to $\alpha_s(M_Z)$-values obtained from high energy experiments.

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The physics of τ-lepton hadronic decays is an important area of particle phenomenology where the theory of strong interaction (QCD) can be confronted with experiment to a very high precision. The central quantity of interest in this process is the spectral density of hadronic states related to the two-point correlator of hadronic currents with well established and simple analytic properties. The accuracy of experimental data for a variety of observables of the τ-lepton system is rather good and is steadily improving [1–3]. The spectral density itself (more precisely, the two-point correlator of hadronic currents in the Euclidean domain) has been calculated with a very high degree of accuracy within perturbation theory (see e.g. [4–7]). Nonperturbative corrections to the correlator are known to be small and under control within the operator product expansion and factorization approximation [8,9]. The observables in the τ system are inclusive in nature which makes the comparison of experimental data with theoretical calculations very clean [10–16]. Of some particular interest is the precise determination of the numerical value of the strong coupling constant at the low energy scale of the τ-lepton mass. Within the renormalization group approach this number can then be evolved to high energies. This is a powerful consistency check of QCD since one is comparing hadron physics at a tremendous variety of scales, from one to hundreds of GeV (e.g. [17]).

In the present note we provide a thorough analysis of the procedure of extracting numerical values of α_s from τ-data in perturbation theory. On the theory side one expects a high degree of accuracy in the determination of α_s because of the existence of very accurate perturbation theory formulas and the simplicity of the renormalization group treatment of the massless quark case. However, the numerical value of the expansion parameter α_s is not small at the M_τ scale and the contribution of higher order terms in the perturbation theory series can be significant. Arguments have been brought forth that the accuracy of finite-order perturbation theory is already close to its asymptotic limit which makes the interpretation (usually called resummation) of the perturbation theory series in higher orders necessary [18]. The resummation of contributions related to the running of the coupling constant is most advanced e.g. [19–25].

The decisive new point of our analysis is the explicit use of renormalization group invariance in the analysis of the τ-lepton decay rate within perturbation theory. Renormalization group invariance is a fundamental property of perturbation theory in quantum field theory which is related to the freedom in defining the subtraction procedure [26]. It should be respected in any numerical analysis. Renormalization group invariance allows one to formally perform the numerical analysis in any renormalization scheme because all schemes are connected by a renormalization group transformation. However, in the finite-order perturbation theory approach this equivalence is only approximate due to the systematic omission of higher order terms in the perturbation theory expressions. This inroduces numerical differences into the results obtained in different renormalization schemes. Generally one can consider two ways of using perturbation theory calculations. One is to find relations between physical observables which are renormalization group invariant. Then perturbation theory calculations are just a purely intermediate step for finding relations between observables (see, e.g. [18,27]) and no numerical analysis for renormalization scheme noninvariant quantities is performed. Indeed, let the perturbation theory expressions for two observables $\mathcal{O}_{1,2}$ in a given scheme have the form

$$\begin{aligned} \mathcal{O}_1 &= \alpha_s + r_1\alpha_s^2 + O(\alpha_s^3), \\ \mathcal{O}_2 &= \alpha_s + r_2\alpha_s^2 + O(\alpha_s^3)\,. \end{aligned} \tag{1}$$

Then the perturbation theory relation between observables $\mathcal{O}_{1,2}$ reads

$$\mathcal{O}_2 = \mathcal{O}_1 + (r_2 - r_1)\mathcal{O}_1^2 + O(\mathcal{O}_1^3) \tag{2}$$

and is scheme-independent. The difference $r_2 - r_1$ takes the same value for calculations in any scheme. Another

way of using perturbation theory calculations is to extract numerical values for renormalization scheme non-invariant quantities (as the coupling constant in a fixed scheme). These are then compared with the results of other experiments. In this case the truncation of the perturbation theory series leads to numerical violations of renormalization scheme invariance and plays an essential role. In our simple example this means that the relations in eq. (1) are treated as quadratic functions of α_s in some fixed scheme and the accuracy of extraction of the coupling constant value (and prediction of other observables) depends drastically on the scheme used, i.e. on the numerical values of the coefficients $r_{1,2}$.

In the present paper we consider just this second application and extract a numerical value for the coupling constant which is not an immediate physical quantity. By convention the reference value of the coupling constant that is used to compare between different experiments is fixed to be the $\overline{\rm MS}$-scheme one. However, and this is our point in this paper, this does not necessarily mean that for its extraction from a given experiment the numerical analysis should be performed in the $\overline{\rm MS}$-scheme. It can be more convenient (and numerically accurate) to analyze the system in its internal scheme and after finding numerical values for the internal parameters translate them into the $\overline{\rm MS}$-scheme using renormalization scheme transformation. This program heavily uses explicit renormalization scheme covariance of the theory. However, expressions for the amplitudes are available only in perturbation theory as a truncated series in the coupling constant. For a truncated series the renormalization scheme invariance is only approximate with a precision of the order of the value of the first omitted term. Therefore numerical values obtained in the $\overline{\rm MS}$-scheme directly and through renormalization group transformations can differ. We discuss this problem and argue that the internal scheme results are most reliable physically and are more stable numerically than the results of the standard analysis in the $\overline{\rm MS}$-scheme. Then numerical values for the reference $\overline{\rm MS}$-scheme parameters can be obtained by a renormalization group "rotation" from the numerical values found in the internal schemes. Renormalization group "rotation" (the re-calculation of numerical values from one scheme to another) is a quite formal operation and can be easily controlled numerically. One example of such a "rotation" (the renormalization group scaling which is a one-parameter subgroup of the general renormalization group) is the evolution of the coupling constant to the reference scale M_Z. Below we give a detailed description of our approach.

The normalized τ-lepton decay rate into nonstrange hadrons $h_{S=0}$ is given by

$$R_{\tau S=0} = \frac{\Gamma(\tau \to h_{S=0}\nu)}{\Gamma(\tau \to l\bar{\nu}\nu)} = N_c|V_{ud}|^2 S_{EW}(1+\delta_P+\delta_{EW}+\delta_{NP}) \qquad (3)$$

where $N_c = 3$ is the number of colors. The first term in eq. (3) is the parton model result while the second term δ_P represents perturbative QCD effects. For the flavor mixing matrix element we use $|V_{ud}|^2 = 0.9511 \pm 0.0014$ [3]. The factor $S_{EW} = 1.0194$ is an electroweak correction term [28] and $\delta_{EW} = 0.001$ is an additive electroweak correction [29]. The nonperturbative corrections are rather small and consistent with zero; we use $\delta_{NP} = -0.003 \pm 0.003$ (see e.g. [13]). Note that recently the problem of duality violation for two-point correlators has been discussed [30,31]. However, no established quantitative estimates of that violation are available yet. Considerations show that they can be rather large and can reach the level of few percents. This problem can affect the numerical value of the coupling extracted from the analysis because of the numerical change of the quantity δ_P extracted from eq. (3). In the present note we concentrate on the perturbative part of the decay rate and numerical uncertainties related to the renormalization scheme freedom of perturbation theory. In this respect new possible corrections do not qualitatively affect our analysis. The corrections due to duality violation are of a new nature and they can be added independently to eq. (3). They would only change the input numerical value for the δ_P within our approach.

The value for the decay rate $R_{\tau S=0}$ has been measured by the ALEPH [1] and OPAL [2] collaborations with results very close to each other. For definiteness we use the ALEPH data and briefly comment on the OPAL data later on. With the experimental result

$$R^{exp}_{\tau S=0} = 3.492 \pm 0.016 \qquad (4)$$

one obtains from eq. (3)

$$\delta^{exp}_P = 0.203 \pm 0.007\ . \qquad (5)$$

The basic object of the theoretical calculation is Adler's D-function which is computable in perturbation theory in the Euclidean domain. In the $\overline{\rm MS}$-scheme the perturbative expansion for the D-function is given by

$$D(Q^2) = 1 + \frac{\alpha_s(Q)}{\pi} + k_1\left(\frac{\alpha_s(Q)}{\pi}\right)^2 + k_2\left(\frac{\alpha_s(Q)}{\pi}\right)^3 + k_3\left(\frac{\alpha_s(Q)}{\pi}\right)^4 + O(\alpha_s(Q)^5) \qquad (6)$$

with (see e.g. [4])

$$k_1 = \frac{299}{24} - 9\zeta(3), \qquad k_2 = \frac{58057}{288} - \frac{779}{4}\zeta(3) + \frac{75}{2}\zeta(5)\,. \qquad (7)$$

Here $\zeta(x)$ is Riemann's ζ-function. In the following we use the notation

$$a_s(Q) = \frac{\alpha_s(Q)}{\pi} \qquad (8)$$

for the standard $\overline{\rm MS}$-coupling constant normalized at the scale $\mu = Q$. Numerically we find

$$D(Q^2) = 1 + a_s(Q) + 1.6398 a_s(Q)^2 + 6.3710 a_s(Q)^3 + k_3 a_s(Q)^4 + O(a_s^5(Q)) \,. \quad (9)$$

The coefficient k_3 is still unknown which prevents us from using the last term in eq. (9) for our analysis. We nevertheless list this term throughout the paper to obtain a feeling for the possible magnitude of the $O(\alpha_s^4)$ correction. The particular numerical value of $k_3 \sim 25$ is obtained on the basis of geometric series approximation for the series (9) and is often used in the literature [16,32,33]. In our analysis we do not use any particular numerical value for k_3 and only give some illustrative results of the influence of this term on the numerical value of the coupling constant extracted from τ-data.

In the $\overline{\rm MS}$-scheme the perturbative correction δ_P is given by the perturbation theory expansion

$$\delta_P^{th} = a_s + 5.2023 a_s^2 + 26.366 a_s^3 + (78.003 + k_3) a_s^4 + O(a_s^5) \quad (10)$$

where the $\overline{\rm MS}$-scheme coupling constant $\alpha_s = \pi a_s$ is taken at the scale of the τ-lepton mass $\mu = M_\tau = 1.777$ GeV. Usually one extracts a numerical value for $\alpha_s(M_\tau)$ by treating the first three terms of the expression in eq. (10) as an exact function – the cubic polynomial, i.e. one solves the equation

$$a_s + 5.2023 a_s^2 + 26.366 a_s^3 = \delta_P^{exp} \,. \quad (11)$$

The solution reads

$$\pi a_s^{st}(M_\tau) \equiv \alpha_s^{st}(M_\tau) = 0.3404 \pm 0.0073_{exp} \,. \quad (12)$$

We call this method the standard method. The quoted error is due to the error in the input value of δ_P^{exp}. We retain some additional decimal points in the numerical expression for the coupling constant in order to use them for the evolution of the coupling constant to the scale M_Z. It is rather difficult to estimate the theoretical uncertainty of the procedure itself. The main problem is to estimate the quality of the approximation for the (asymptotic) series in eq. (10) given by the cubic polynomial in eq. (11).

As a criterion of the quality of the approximation one can use the pattern of convergence of the series (10) which is

$$\delta_P^{exp} = 0.203 = 0.108 + 0.061 + 0.034 + \ldots \quad (13)$$

One sees that the corrections provide a 100% change of the leading term. Another criterion is the order-by-order behavior of the extracted numerical value for the coupling constant. In consecutive orders of perturbation theory (LO - leading order, NLO - next-to-leading order, NNLO - next-next-to-leading order) one has

$$\alpha_s^{st}(M_\tau)_{LO} = 0.6377, \quad \alpha_s^{st}(M_\tau)_{NLO} = 0.3882, \quad \alpha_s^{st}(M_\tau)_{NNLO} = 0.3404 \,. \quad (14)$$

Formally we obtain a series for the numerical value of the coupling constant of the form

$$\alpha_s^{st}(M_\tau)_{NNLO} = 0.6377 - 0.2495 - 0.0478 - \ldots \quad (15)$$

Limiting ourselves to the next-to-next-to-leading order result (NNLO) we can take a half of the last term as an estimate of the theoretical uncertainty. It is only an indicative estimate. No rigorous justification can be given for such an assumption about the accuracy of the approximation without knowledge of the structure of the whole series. Nevertheless we stick to this definition for our purposes. The theoretical uncertainty obtained in such a way – $\Delta\alpha_s^{st}(M_\tau)_{th} = 0.0239$ – is much larger than the experimental uncertainty given in eq. (12). This is a challenge for the theory: the accuracy of theoretical formulas cannot compete with experimental precision at present. Assuming this theoretical uncertainty we have

$$\alpha_s^{st}(M_\tau)_{NNLO} = 0.3404 \pm 0.0239_{th} \pm 0.0073_{exp}. \quad (16)$$

Theory dominates the error even if the estimate for its precision $\pm 0.0239_{th}$ is not reliable (heuristic and only indicative). Thus the straightforward analysis in the $\overline{\rm MS}$-scheme is not stable numerically and the naive estimate of the theoretical uncertainty is large.

The use of the $\overline{\rm MS}$-scheme is not obligatory for practical calculations. The $\overline{\rm MS}$-scheme has a history of success for massless calculations where its results look natural and the corrections are usually small. This is not the strict rule, however, and there are cases (like gluonic correlators [34]) where corrections dramatically depend on the quantum numbers of the operators. In fact, the $\overline{\rm MS}$-scheme is rather artificial. It is simply defined by convention (let us be remindful of the evolution from the MS-scheme to the $\overline{\rm MS}$-scheme which had its origin only in technical convenience [35]). From technical point of view, in practical calculations of massless diagrams of the propagator type, another scheme – the G-scheme – is the most natural one [36]. It normalizes the basic quantity of the whole calculation within integration-by-parts technique – one loop masless scalar diagram – to unity [37]. β-functions coincide in both schemes. It could have well happened that the G-scheme would be historically adopted as the reference scheme because corrections in this scheme are typically smaller than that in the $\overline{\rm MS}$-scheme. However, for the tau system the direct (standard) analysis in the G-scheme fails.

Therefore different schemes used for the numerical analysis can produce rather different numerical results for the final reference quantity - the coupling constant in the $\overline{\rm MS}$-scheme. Note that strictly speaking any scheme is suitable for a given perturbative calculation. However, it can lead to unusual (or even unacceptable) results in a numerical analysis. The only criterion for the choice of scheme at present is the heuristic requirement of fast

explicitl convergence: the terms of the series should decrease. Clearly this is a rather unreliable criterion. It does not provide strict quantitative constraints necessary for the level of precision usually claimed for the τ-system analysis.

In the following we suggest a new procedure for extracting α_s in the $\overline{\rm MS}$-scheme from the τ system without explicit use of eq. (10). This procedure is applicable to any observable in whatever scheme it was originally computed. The observation is that any perturbation theory observable generates a scale due to dimensional transmutation and this is its internal scale. It is natural for a numerical analysis (and is our suggestion) to determine this scale fisrt and then to transform the result into a $\overline{\rm MS}$-scheme parameter (or any other reference scheme) using the renormalization group invariance. We deliberately use the explicit renormalization scheme invariance of the theory to bring the result of the perturbation theory calculation into a special scheme first, then we perform a numerical analysis in this particular scheme. Only after that we transform the obtained numbers into the reference $\overline{\rm MS}$-scheme. The last step is done only for comparison with other experiments (or just for convenience; the system itself can be well described in its internal scheme without any reference to the $\overline{\rm MS}$-scheme). This is our suggestion for the resolution of the problem of numerical instability of extracting parameters from truncated perturbation theory expressions.

A dimensional scale in QCD emerges as a boundary value parameterizing the evolution trajectoty of the coupling constant. The renormalization group equation

$$\mu^2\frac{d}{d\mu^2}a(\mu^2)=\beta(a(\mu^2))\,,\quad a=\frac{\alpha}{\pi} \tag{17}$$

is solved by the integral

$$\ln\left(\frac{\mu^2}{\Lambda^2}\right)=\Phi(a(\mu^2))+\int_0^{a(\mu^2)}\left(\frac{1}{\beta(\xi)}-\frac{1}{\beta_2(\xi)}\right)d\xi \tag{18}$$

where the indefinite integral $\Phi(a)$ is normalized as follows

$$\Phi(a)=\int^a\frac{1}{\beta_2(\xi)}d\xi=\frac{1}{a\beta_0}+\frac{\beta_1}{\beta_0^2}\ln\left(\frac{a\beta_0^2}{\beta_0+a\beta_1}\right). \tag{19}$$

Here $\beta_2(a)$ and $\beta(a)$ denote the second order and full β function, or as many terms as are available, given by

$$\begin{aligned}\beta_2(a)&=-a^2(\beta_0+a\beta_1),\\ \beta(a)&=-a^2(\beta_0+\beta_1a+\beta_2a^2+\beta_3a^3)+O(a^6)\,,\end{aligned} \tag{20}$$

a is a generic coupling constant. The four-loop β-function coefficient β_3 is now known in the $\overline{\rm MS}$-scheme [38]

$$\beta_3=\frac{140599}{4608}+\frac{445}{32}\zeta(3)=47.228\ldots \tag{21}$$

The integration constant in eq. (18) is adjusted such that the asymptotic expansion of the coupling constant at large momenta $Q^2\to\infty$ reads

$$\begin{aligned}a(Q^2)&=\frac{1}{\beta_0L}\left(1-\frac{\beta_1}{\beta_0^2}\frac{\ln(L)}{L^2}\right)+O\left(\frac{1}{L^3}\right)\,,\\ L&=\ln\left(\frac{Q^2}{\Lambda^2}\right)\,.\end{aligned} \tag{22}$$

This serves to define the parameter Λ (dimensional scale) for a generic coupling constant. Λ_s is the standard $\overline{\rm MS}$-scheme scale for the coupling constant a_s.

The solution (18) of the renormalization group equation (17) describes the evolution trajectory of the coupling constant. This trajectory is parametrized by the scale parameter Λ and the coefficients of the β function β_i with $i>2$ (see e.g. [39]). The evolution is invariant under the renormalization group transformation

$$a\to a(1+\kappa_1a+\kappa_2a^2+\kappa_3a^3+\ldots) \tag{23}$$

with the simultaneous change

$$\Lambda^2\to\Lambda^2e^{-\kappa_1/\beta_0}\,, \tag{24}$$

$\beta_{0,1}$ left invariant and

$$\begin{aligned}&\beta_2\to\beta_2-\kappa_1^2\beta_0+\kappa_2\beta_0-\kappa_1\beta_1\\ &\beta_3\to\beta_3+4\kappa_1^3\beta_0+2\kappa_3\beta_0+\kappa_1^2\beta_1-2\kappa_1(3\kappa_2\beta_0+\beta_2).\end{aligned}$$

If this transformation was considered to be exact and the exact β-function corresponding to the new charge was used then it would be just a change of variable in a differential equation (17) or the exact reparametrization of the trajectory (18) and hence would lead to identical results. However, the renormalization group invariance of eq. (18) is violated in higher orders of the coupling constant because we consistently omit higher orders in the perturbation theory expressions for the β-functions. This is the point where the finite-order perturbation theory approximation for the respective β-functions is made. This is the source for different numerical outputs of analyses in different schemes.

Our procedure for the extraction of α_s is heavily based on the formal renormalization group invariance of the theory. We claim that because of this invariance we can do our numerical analysis in any scheme. The reason for the choice of a particular scheme is only the quality of the convergence (which, of course, is subject to some personal taste). We have chosen the effective scheme because we consider it to be more consistent and more stable numerically.

Technically we introduce an effective charge a_τ through the relation [27,40–43]

$$\delta_P^{th}=a_\tau\equiv\frac{\alpha_\tau}{\pi} \tag{25}$$

and extract the parameter Λ_τ which is associated with a_τ through eq. (18). This is just the internal scale associated with the physical observable R_τ. The effective β-function is given by the expression

$$\beta_\tau=-a_\tau^2(\beta_{\tau0}+\beta_{\tau1}a_\tau+\beta_{\tau2}a_\tau^2+\beta_{\tau3}a_\tau^3+\ldots) \tag{26}$$

with $\beta_{\tau 0}=\beta_0$, $\beta_{\tau 1}=\beta_1$, and

$$\beta_{\tau 2}=-12.3204\,,\quad \beta_{\tau 3}=-182.719+\frac{9}{2}k_3\,. \tag{27}$$

The extraction of the numerical value for the internal scale Λ_τ is done from equation (18) with $a_\tau(M_\tau)=\delta_P^{exp}$. The coefficient $\beta_{\tau 3}$ does not enter the analysis. The parameter $\Lambda_s\equiv\Lambda_{\overline{\rm MS}}$ is found according to eq. (24). The $\overline{\rm MS}$ coupling at $\mu=M_\tau$ is obtained by solving eq. (18) for $a_s(M_\tau)$ with regard to $\ln(M_\tau^2/\Lambda_s^2)$ which is known if Λ_s is obtained; the β-function is taken in the $\overline{\rm MS}$-scheme. For consistency reasons we only use the $\overline{\rm MS}$-scheme β-function to three-loop order since the effective β-function β_τ is only known up to the second order, cf. eq. (27). A N^3LO analysis is possible only if a definite value is chosen for k_3. We give some estimates later.

Our procedure is based on renormalization group invariance and one can start from the expression for the decay rate obtained in any scheme. The only perturbative objects present are the β-functions. Both $\beta_{\overline{\rm MS}}$ and β_τ, however, converge reasonably well which is the only perturbation theory restriction in our method. It also highlights the limit of precision within our procedure: the expansion for β_τ is believed to be asymptotic as any expansion in perturbation theory. The asymptotic expansion provides only limited accuracy for any given numerical value of the expansion parameter which cannot be further improved by including higher order terms. The expansion used is presumably rather close to its asymptotic limit as can be seen by taking a look at the expansion

$$\beta_\tau(a_\tau)=-a_\tau^2\Big(\frac{9}{4}+4a_\tau-12.3204a_\tau^2 +a_\tau^3\left(-182.719+\frac{9}{2}k_3\right)\Big)+O(a_\tau^6) \tag{28}$$

with $a_\tau\sim 0.2$ at the scale M_τ. The convergence of the series depends crucially on the numerical value of k_3. If k_3 had a value where the asymptotic growth starts at third order then further improvement of the accuracy within finite-order perturbation theory is impossible.

At every order of the analysis we use the whole information of the perturbation theory calculation. Especially, the appropriate coefficient of the β_τ-function is present. In the standard method the coefficient β_2 enters only at order $O(\alpha_s^4)$ of the τ-lepton decay rate expansion. We call our procedure the renormalization scheme invariant extraction method (RSI) hoping that it is clear what is meant by this name from our explanations. Note also that α_s itself is not a physical object and is renormalization scheme noninvariant. In this respect we extract the noninvariant parameter α_s using invariance of the physics in order to perform the numerical analysis in the most suitable scheme. Then the output of the analysis is simply transformed into a numerical value for α_s according to the renormalization group transformation rules. For the coupling constant in the $\overline{\rm MS}$-scheme in NNLO we find

$$\alpha_s^{RSI}(M_\tau)=0.3184\pm 0.0060_{exp} \tag{29}$$

which is smaller than the corresponding value obtained within the standard procedure eq. (12). How to estimate the quality of this result? The parameter which is really extracted in consecutive orders of perturbation theory within our method is the scale Λ_τ. Because of the relation (see eqs. (10,23,24))

$$\Lambda_s=\Lambda_\tau e^{-5.20232/2\beta_0}=0.3147\Lambda_\tau \tag{30}$$

we can look at Λ_s directly. We find

$$\Lambda_s|_{LO}=595\ \text{MeV},\quad \Lambda_s|_{NLO}=288\ \text{MeV},\quad \Lambda_s|_{NNLO}=349\ \text{MeV} \tag{31}$$

or, representing the NNLO result as a formal series,

$$\Lambda_s|_{NNLO}=595-307+61-\ldots\ \text{MeV}. \tag{32}$$

Note that at leading order the scales (as well as charges) are equal in all schemes. Therefore the leading order result ($\Lambda_s|_{LO}=595$ MeV) is not representative, only indicative. Assuming according to our convention that the uncertainty of Λ_s is given by the half of the last term of the series (32) we have

$$\Lambda_s=349\pm 31\ \text{MeV} \tag{33}$$

which leads to the numerical value for the $\overline{\rm MS}$-scheme coupling constant

$$\alpha_s=0.3184^{-0.0157}_{+0.0160}\,. \tag{34}$$

This result is obtained from eq. (18) with three-loop β-function. Taking the average we find

$$\alpha_s=0.3184\pm 0.0159\,. \tag{35}$$

This is better than the theoretical error of the standard result eq. (16). Still the theoretical error should be considered as a guess rather than a well-justified estimate of the uncertainty.

Let us briefly comment on the k_3 contribution. Clearly the estimate $k_3=25$ is rather speculative. We, therefore, use a different strategy in the analysis. We determine the range of k_3 which is safe for explicit convergence of perturbation theory. If the actual value of k_3 will be discovered in this range then perturbation theory is still valid and will give better accuracy in NNNLO. If not, the asymptotic growth of perturbation theory series is already reached and its accuracy cannot be improved.

We require that the last term is equal to the half of the previous one. In the standard way (eq. 10) we have

$$|(78+k_3)a_s|<\frac{1}{2}\,26.36\approx 13 \tag{36}$$

which for $a_s=0.1$ gives

$$-208<k_3^{st}<52\,. \tag{37}$$

In the RSI way (eq. 28) we have

$$|(-182+\frac{9}{2}k_3)a_\tau| < \frac{1}{2}\,12.32 \approx 6 \qquad (38)$$

which for $a_\tau = 0.2$ gives

$$33.8 < k_3^\tau < 47.1\,. \qquad (39)$$

This range is much narrower than that in eq. (37). The effective scheme method is much more sensitive to the structure of the series as can be seen from eq. (28). The actual precision depends on the actual value chosen for k_3 and it is rather premature to speculate about numbers.

Still we show the worst result (in the optimistic scenario that k_3 lies in the safe range) that can be expected within the RSI approach. In the RSI approach with $k_3 = 47$ we find the scale parameter in NNNLO

$$\Lambda_s|_{NNNLO} = 334 \text{ MeV}\,. \qquad (40)$$

With $k_3 = 34$ one has

$$\Lambda_s|_{NNNLO} = 367 \text{ MeV}\,. \qquad (41)$$

Taking the average we have

$$\Lambda_s = 350 \pm 17 \text{ MeV} \qquad (42)$$

which is the best possible estimate if we require that the perturbation theory series for the β_τ-function still converges (according to our quantitative criterion of convergence). That results in the numerical value for the $\overline{\text{MS}}$-scheme coupling constant found with four-loop β-function from eq. (18)

$$0.3133 < \alpha_s < 0.3314\,. \qquad (43)$$

Therefore our conservative estimate of the theoretical error in the optimistic scenario for the convergence of perturbation theory series in NNNLO reads

$$\alpha_s = 0.322 \pm 0.009\,. \qquad (44)$$

While the estimation of the theoretical uncertainty is a tricky matter and can be considered as indicative the central numerical value of the coupling constant definitely becomes smaller as compared to the standard result.

At present the reference value for the coupling constant is commonly given at the scale $M_Z = 91.187$ GeV. The running to this reference scale is done with the four-loop β-function in the $\overline{\text{MS}}$-scheme [38] and three-loop matching conditions at the heavy quark (charm and bottom) thresholds [44]. For the threshold parameters related to heavy quark masses we use $\mu_c = \bar{m}_c(\mu_c) = (1.35 \pm 0.15)$ GeV and $\mu_b = \bar{m}_b(\mu_b) = (4.21 \pm 0.11)$ GeV (e.g. [45]) where $\bar{m}_q(\mu)$ is the running mass of the heavy quark in the $\overline{\text{MS}}$-scheme. Note that because of the truncation of matching conditions the result of the running slightly depends on at what scale the matching is actually performed. If the matching between the $n_f = 3$ and $n_f = 4$ effective theories is done directly at the scale M_τ, which is possible, then the result is slighly smaller than in the case when the evolution within $n_f = 3$ effective theory is done first to the scale μ_c. In the following we stick to the procedure where the matching is performed precisely at the matching scales $\mu_{c,b}$. We first run the coupling constant within $n_f = 3$ effective theory from the scale M_τ to μ_c then match the result to $n_f = 4$ coupling constant, run it to μ_b and match to $n_f = 5$ coupling constant. The last step is just evolution to M_Z. Note that the alternative would be to perform matching between $n_f = 3$ and $n_f = 4$ effective theories directly at the scale M_τ (because it is rather close to μ_c) but in this case the final result is slightly smaller than in our present procedure.

The running to the scale M_Z gives the following result for the standard method estimate

$$\alpha_s^{st}(M_Z) = 0.1210 \pm 0.0008_{exp} \pm 0.0006_c \pm 0.0001_b \quad (45)$$

where the subscript exp denotes the error originating from δ_P^{exp}. The errors with subscripts c, b arise from the uncertainty of the numerical values of the charm and bottom quark masses that enter the evolution analysis. These errors are rather small (we retain the additional decimal place in the result, which is not really justified from the precision of the experimental input, just to show these uncertainties). If the matching between the $n_f = 3$ and $n_f = 4$ effective theories is done directly at the scale M_τ one has to change the central value $0.1210 \to 0.1202$ which shows the uncertainty related to the truncation of the matching conditions.

The central value in eq. (45) is slightly higher than that calculated from high energy experiments [3]. The theoretical perturbative expansions for observables in high energy experiments converge better numerically than expansions at low energies because the coupling, which is the parameter of the perturbative expansion, is smaller at higher energies due to the property of asymptotic freedom. This feature makes it less important to treat the higher order terms carefully in high energy applications as compared to the low energy τ-lepton estimates. However, the experimental data in high energy experiments are usually less precise which leads to large errors in the α_s determination from high energy experiments. The fact that the value in eq. (45) is higher than that calculated from high energy experiments caused some discussion about the reliability of estimates from the τ-lepton data. Our analysis resolves this problem. The running of $\alpha_s^{RSI}(M_\tau)$ given in eq. (35) to M_Z with the four-loop β-function and with three-loop heavy quark matching accuracy gives

$$\alpha_s^{RSI}(M_Z) = 0.1184 \pm 0.00074_{exp} \pm 0.00053_c \pm 0.00005_b \qquad (46)$$

where we have kept five decimal places in order to exhibit the magnitude of different sources of uncertainty.

Eq. (46) constitutes our main result for the coupling $\alpha_s(M_Z)$ derived from tau data.

The OPAL collaboration has reported an experimental value of $R^{exp}_{\tau S=0} = 3.484 \pm 0.024$ [2]. This leads to $\delta^{exp}_P = 0.200 \pm 0.009_{exp}$ and

$$\alpha_s^{RSI}(M_\tau) = 0.3158 \pm 0.0078_{exp} \tag{47}$$

which, when evolved to M_Z, gives

$$\alpha_s^{RSI}(M_Z) = 0.1181 \pm 0.00097_{exp} \pm 0.00052_c \pm 0.00005_b\,. \tag{48}$$

This value is close to the one in eq. (46) based on the ALEPH data.

The theoretical uncertainty comes mainly from the truncation of the perturbation theory series. Taking the result of the NNLO analysis eq. (35) we find

$$\Delta\alpha_s^{RSI}(M_Z)_{th} = 0.0019 \tag{49}$$

In the most optimistic scenario with the NNNLO analysis eq. (44) one has

$$\alpha_s^{RSI}(M_Z)_{N^3LO} = 0.119 \pm 0.001\,. \tag{50}$$

As we have already noted the interpretation of the higher order terms in the perturbation theory expansion is numerically important for the analysis of the τ-data. The regular method to resum higher order perturbation theory corrections is based on the direct integration of the renormalization group improved correlators over the contour in the complex Q^2 plane [14]. This method allows one to resum corrections generated by the running of the coupling constant along the integration contour and is now widely used for the analysis of the τ-data. We now briefly comment on the extraction of the strong coupling constant within resummed perturbation theory. As in ref. [46] we fit the theoretical expression for the decay rate in the contour improved approach to the experimental result δ^{exp}_P eq. (5) and find

$$\alpha_s^{CI}(M_\tau) = 0.343 \pm 0.009_{exp} \tag{51}$$

within the renormalization scheme invariant extraction method described above i.e. with the introduction of the effective charge first. This value differs from the finite-order perturbation theory result eq. (29). Note that the two values extracted from finite-order perturbation theory analysis eq. (29) and the contour improved perturbation theory analysis eq. (51) do not overlap within their respective error bars given from the experimental uncertainty only. This situation was anticipated in [14] where the resummed NNLO analysis had been first performed. The point is clear: resummation provides a specific estimate of higher order terms. In finite-order perturbation theory one adopts a model where all higher order terms have been neglected. In contour improved perturbation theory one adopts an explicit model with higher order terms generated by the running of the coupling constant along the integration contour. With present experimental accuracy one can already distinguish between these two possibilities. One should always keep in mind that the two determinations eq. (29) and eq. (51) result from different models and one should not mix their predictions.

The numerical value of the coupling constant appropriate for high energy experiments is normally small (much smaller than for τ-data) and perturbation theory converges faster (in similar kinematical situations). The resummation does not produce any big numerical changes. Therefore finite-order perturbation theory is normally used for the analysis of high energy experiments (resummation of the contour type can be done but produces a small numerical effect) and one usually quotes numerical values of the coupling constant extracted with finite-order perturbation theory. Or resummation of the sort different from that used for the τ system is used (like Coulomb type resummation for heavy quarks [47,48]). Therefore we suggest to use the finite-order perturbation theory prediction for the coupling constant extracted from τ-data in order to compare it with the results of high energy experiments.

To conclude, we have extracted the numerical value of the strong coupling constant from τ-data within a procedure based on explicit use of renormalization scheme invariance. The numerical value for the coupling constant is systematically smaller than that derived by the standard treatment. When evolved to M_Z our $\overline{\rm MS}$-scheme value for the coupling constant extracted in finite-order perturbation theory reads

$$\alpha_s(M_Z) = 0.1184 \pm 0.0007_{exp} \pm 0.0006_{hq\ mass}\,. \tag{52}$$

This central value is closer to the value of α_s derived from high energy experiments than previous determinations of α_s from τ-data. The theoretical uncertainty of the result is still only indicative: it ranges from the conservative estimate in NNLO $\Delta\alpha_s(M_Z)_{th} = \pm 0.0019$ to an optimistic one based on the assumption about NNNLO contribution $\Delta\alpha_s(M_Z)_{th} = \pm 0.001$.

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